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The Potential in a Cylindrical Electrode Partially Submerged in a Perfectly Conducting Medium[1]

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ABSTRACT

A method, well suited for use with digital computers, is given for determining the potential in a cylindrical electrode partially submerged in a conducting medium. Numerical illustrations and a brief proof of the convergence of the approximants are provided.

I. INTRODUCTION

In an early application of the method of dual integral equations Danilevsky [1] studied the potential in an infinite cylindrical electrode partially submerged in a perfectly conducting medium. That is, Danilevsky sought the function u satisfying the following conditions.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0, \qquad -\infty < x < \infty; \quad 0 < r < 1, \tag{1}$$

$$u = 0, \qquad x < 0; \quad r = 1, \tag{2}$$

$$\frac{\partial u}{\partial r} = 0, \qquad x > 0; \quad r = 1, \tag{3}$$

$$\frac{\partial u}{\partial x} = E_0, \qquad \text{as} \quad x \to \infty, \tag{4}$$

$$u = 0, \qquad \text{as} \quad x \to -\infty, \tag{5}$$

where E_0 is a given constant.

Danilevsky's method has subsequently been applied [2–4] to other problems for Laplace's equation in infinite cylinders with broken boundary conditions. Since the method depends, in essence, on an ingenious application of the Fourier transform, it does not readily extend to finite, semi-infinite, and composite

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cylinders. In this paper we present a general procedure for all these configurations which is well adapted for use with modern computers. A theoretical advantage of this procedure, compared to the methods of multiple series and multiple integral equations, cf. [5], is that it allows for a very short, but mathematically rigorous, proof of the existence of the Fourier-Bessel coefficients and the convergence of approximants to these coefficients (Detailed regularity arguments [6; 7] require more delicate and extensive analysis, but are of little interest for applications).

II. Solutions

Let J_0 and J_1 be Bessel functions of the first kind. γ_k and α_k $(k = 1, 2,...)$ denote respectively the kth nonnegative roots of J_0 and J_1. Note that $\alpha_1 = 0$. We denote by $\mathscr{J}_0(\gamma_k, r, 1)$ and $\mathscr{J}_0(\alpha_k, r, 0)$, $k = 1, 2,...$, (abbreviated hereafter to $\mathscr{J}_0(\gamma_k r)$ and $\mathscr{J}_0(\alpha_k r)$) the normalized Bessel functions

$$\mathscr{J}_0(\gamma_k r) = \frac{\sqrt{2}\, J_0(\gamma_k r)}{J_1(\gamma_k)}, \qquad \mathscr{J}_0(\alpha_k r) = \frac{\sqrt{2}\, J_0(\alpha_k r)}{J_0(\alpha_k)}.$$

By ℓ^2 we denote the real Hilbert space of all square summable infinite column vectors $\mathfrak{r} = (\mathfrak{r}_1, \mathfrak{r}_2, ...)$ formed with inner product $(\mathfrak{r}, \mathfrak{s}) = \Sigma \mathfrak{r}_n \mathfrak{s}_n$ and the norm $\| \mathfrak{r} \| = (\mathfrak{r}, \mathfrak{r})^{1/2}$. The domain of an operator V on ℓ^2 is denoted by D_V.

We begin by seeking vectors $\mathfrak{j}$ and $\mathfrak{p}$ and a solution u given by

$$u(x, r) = \sum_{k=1}^{\infty} \mathfrak{j}_k \mathscr{J}_0(\gamma_k r)\, e^{\gamma_k x}, \qquad x < 0; \quad 0 \leqslant r \leqslant 1,$$

$$u(x, r) = \sum_{k=1}^{\infty} \mathfrak{p}_k \mathscr{J}_0(\alpha_k r)\, e^{-\alpha_k x} + E_0 x, \qquad x > 0; \quad 0 \leqslant r \leqslant 1.$$

We set

$$\lim_{\xi = 0+} \int_0^1 \mathscr{J}_0(\alpha_k r)\,(u(\xi, r) - u(-\xi, r))\, r\,dr = 0, \tag{6}$$

$$\lim_{\xi = 0+} \int_0^1 \mathscr{J}_0(\gamma_k r) \left(\frac{\partial u}{\partial x}\bigg|_{x=\xi} - \frac{\partial u}{\partial x}\bigg|_{x=-\xi} \right) r\,dr = 0, \; k = 1, 2, \ldots. \tag{7}$$

This yields the equations

$$\mathfrak{p} = B\mathfrak{j} \qquad \text{and} \qquad \mathfrak{j} = A\mathfrak{p} + E_0 \mathfrak{b} \tag{8}$$

where the vector $\mathfrak{b}$ and the matrices $A = (A_{kn})$ and $B = (B_{kn})$ are given by $\mathfrak{b}_k = \sqrt{2}\,\gamma_k^{-2}$ and

$$A_{kn} = \frac{-2\alpha_n}{\gamma_k^2 - \alpha_n^2}, \qquad B_{kn} = \frac{2\gamma_n}{\gamma_n^2 - \alpha_k^2}, \qquad k, n = 1, 2, \ldots .$$

Set $D = -\gamma^{1/2}AB\gamma^{-1/2}$ where $\gamma = (\gamma_{kn})$ is the diagonal matrix $\gamma_{kk} = \gamma_k$. If we put $\mathfrak{j} = \gamma^{-1/2}\mathfrak{m}$, we obtain

$$(I + D)\mathfrak{m} = E_0\gamma^{1/2}\mathfrak{b} \tag{9}$$

where I is the identity matrix and

$$D_{kn} = 4(\gamma_k\gamma_n)^{1/2} \sum_{j=2}^{\infty} \frac{\alpha_j}{(\gamma_k^2 - \alpha_j^2)(\gamma_n^2 - \alpha_j^2)}$$

Let the vectors $\{\mathfrak{e}(n) : n = 1, 2, \ldots\}$, defined by $\mathfrak{e}_k(n) = \delta_{kn}$ (δ_{kn} is the Kronecker delta), be the coordinates of the (Rayleigh) Ritz approximation procedure [8, Sec. 8]. In other words for each positive integer N we seek an approximation vector $\mathfrak{m}(N)$ satisfying

$$\mathfrak{m}_k(N) + \sum_{n=1}^{N} D_{kn}\mathfrak{m}_n(N) = E_0\gamma_{kk}^{1/2}\mathfrak{b}_k, \qquad k = 1, 2, \ldots N, \tag{10}$$

with the convention that $\mathfrak{m}_k(N) = 0$ for $k \geqslant N + 1$.

A Fortran IV program using (10) was written by the author. This program computes $\mathfrak{m}(N)$, $\mathfrak{j}(N)$, $\mathfrak{p}(N)$ and u for given $N \leqslant 40$. The coefficients D_{kn} were computed to 6 decimal place accuracy using the first 60 terms in the series giving D_{kn} plus a correction term for the remainder. The roots α_k and γ_k were taken from [9], and J_0 and J_1 were computed from the polynomial approximations in [10]. The system (10) was solved by Gaussian elimination with iteration on the residues. Then $\mathfrak{p}$ was computed from the first equation in (8). For all practical purposes the approximation $N = 20$ seems accurate enough. We found that

$$\frac{\| \mathfrak{j}(40) - \mathfrak{j}(20) \|}{\| \mathfrak{j}(40) \|} \quad \text{and} \quad \frac{\| \mathfrak{p}(40) - \mathfrak{p}(20) \|}{\| \mathfrak{p}(40) \|}$$

were less than 0.04. As an illustration of the results obtained, we give in Table 1 the vectors $\mathfrak{j}$ and $\mathfrak{p}$ for two configurations. In Fig. 1 the lines of equipotential are shown for the infinite composite cylinder (see Sec. 3).

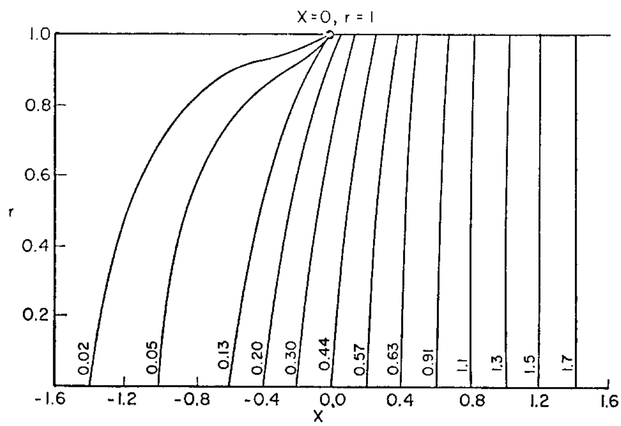

FIG. 1. Lines of Equipotential for $u(x, r)$ in the infinite cylinder. $\sigma_1 = 1.0$, $\sigma_3 = 0.5$, $E_0 = 1.0$.

TABLE I

THE COEFFICIENTS $\mathfrak{j}$ AND $\mathfrak{p}$ FOR THE INFINITE CYLINDER AND FOR THE FINITE CYLINDER WITH $\lambda = 0.1$ AND $\mu = 1.0$[a]

	Infinite cylinder		Finite cylinder	
			$\lambda = 0.1$	$\mu = 0.1$
n	$\mathfrak{j}_n$	$\mathfrak{p}_n$	$\mathfrak{j}_n$	$\mathfrak{p}_n$
1	0.185	0.184	0.00919	0.00946
2	.047	−.053	.00285	−.00208
3	.023	−.023	.00148	−.00113
4	.014	−.013	.00093	−.00075
5	.010	−.009	.00065	−.00055
6	.008	−.007	.00048	−.00043
7	.006	−.005	.00037	−.00035
8	.005	−.004	.00030	−.00029
9	.004	−.003	.00025	−.00025
10	.003	−.003	.00021	−.00022

[a] $E_0 = 1.0$ in both cases.

III. Other Configurations

Let λ and μ be positive numbers. For the finite cylinder we seek u such that Eq. (1) holds for $-\lambda < x < \mu$; Eq. (2) holds for $-\lambda < x < 0$; Eq. (3) holds for $\mu > x > 0$; Eq. (4) holds for $x = \mu$ and $0 < r < 1$; Eq. (5) holds for $x = -\lambda$ and $0 < r < 1$. The solution is given by

$$u = \sum_{k=1}^{\infty} \mathfrak{j}_k \frac{\sinh \gamma_k(x+\lambda)\, \mathscr{J}_0(\gamma_k r)}{\sinh \gamma_k \lambda}, \qquad -\lambda \leqslant x < 0; \quad 0 \leqslant r \leqslant 1,$$

$$u = \sum_{k=1}^{\infty} \mathfrak{p}_k \frac{\cosh \alpha_k(x-\mu)\, \mathscr{J}_0(\alpha_k r)}{\cosh \alpha_k \mu} + E_0 x, \qquad 0 < x \leqslant \mu; \quad 0 \leqslant r \leqslant 1,$$

where $\mathfrak{j}$ and $\mathfrak{p}$ are now determined by

$$\mathfrak{p} = B\mathfrak{j}, \qquad \mathfrak{j} = (\gamma t)^{-1/2}\mathfrak{m}, \qquad (I+F)\mathfrak{m} = E_0(\gamma t)^{1/2}\mathfrak{b}$$

where the diagonal matrix t is given by $t_{kk} = \coth \gamma_k \lambda$ and F is given by

$$F_{kn} = 4(\gamma_k \gamma_n \tanh \gamma_k \lambda \tanh \gamma_n \lambda)^{1/2} \sum_{j=2}^{\infty} \frac{\alpha_j \tanh \alpha_j \mu}{(\gamma_k^2 - \alpha_j^2)(\gamma_n^2 - \alpha_j^2)}.$$

A similar result can be obtained for the semi-infinite cylinder.

If the material in the infinite cylinder has conductivity σ_1 for $x < 0$ and σ_3 for $x > 0$ we replace (7) by

$$\lim_{\xi = 0+} \int_0^1 \mathscr{J}_0(\gamma_k r) \left(\sigma_3 \frac{\partial u}{\partial x}\bigg|_{x=\xi} - \sigma_1 \frac{\partial u}{\partial x}\bigg|_{x=-\xi} \right) r dr = 0.$$

Upon replacement of A by $(\sigma_3/\sigma_1)\, A$ the analysis can be completed exactly as above.

IV. Existence and Convergence

We wish to prove the following.

Theorem. *Equations* (8) *and* (9) *possess a unique solution* $\mathfrak{j}, \mathfrak{p}$, *and* $\mathfrak{m}$ *in* ℓ^2. *Equation* (10) *possesses a unique solution* $\mathfrak{m}(N)$ *such that* $\| \mathfrak{m}(N) - \mathfrak{m} \| \to 0$ *as* $N \to \infty$.

Proof. We observe that $D = TT^*$ where

$$T_{kn} = \frac{2(\gamma_k \alpha_n)^{1/2}}{\gamma_k^2 - \alpha_n^2},$$

and T^* is the transpose of T. Let Δ be the diagonal matrix $\Delta_{kk} = T_{kk}$. Clearly $D_\Delta = \ell^2$. Let L be the matrix given by $L_{kk} = 0$

$$L_{kn} = \frac{2(kn)^{1/2}}{\pi(k^2 - n^2)}, \qquad k \neq n.$$

Schur [11] has shown that $D_L = \ell^2$. It follows using standard asymptotic results for the roots of Bessel functions that $T - \Delta = L + G$ where G is such that $\Sigma_{kn} G_{kn}^2 < \infty$. Therefore $D_G = \ell^2$. Thus $I + D$ is a positive definite symmetric operator of domain ℓ^2. Hence [8], Eqs. (9) and (10) possess unique solutions in ℓ^2 and $\| \mathfrak{m}(N) - \mathfrak{m} \| \to 0$ as $N \to \infty$. Consequently $\mathfrak{j}$ is also uniquely determined. Since $\{\mathscr{J}_0(\alpha_k r)\}$ and $\{\mathscr{J}_0(\gamma_k r)\}$ are bases in the Hilbert space of functions f such that rf^2 is Lebesgue integrable on $0 \leqslant r \leqslant 1$, it follows that B is a unitary operator. Therefore $\mathfrak{p}$ is a uniquely determined element of ℓ^2.

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